

Available online at www.sciencedirect.com**ScienceDirect**

Nuclear Physics B 886 (2014) 1128–1152

**NUCLEAR
PHYSICS B**www.elsevier.com/locate/nuclphysb

Maxwell superalgebras and Abelian semigroup expansion

P.K. Concha^{a,b,c}, E.K. Rodríguez^{a,b,c}^a *Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile*^b *Dipartimento di Scienza Applicata e Tecnologia (DISAT), Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy*^c *Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Torino, Via Pietro Giuria, 1, 10125 Torino, Italy*

Received 22 May 2014; received in revised form 3 July 2014; accepted 22 July 2014

Available online 24 July 2014

Editor: Stephan Stieberger

Abstract

The Abelian semigroup expansion is a powerful and simple method to derive new Lie algebras from a given one. Recently it was shown that the S -expansion of $\mathfrak{so}(3, 2)$ leads us to the Maxwell algebra \mathcal{M} . In this paper we extend this result to superalgebras, by proving that different choices of abelian semigroups S lead to interesting $D = 4$ Maxwell Superalgebras. In particular, the minimal Maxwell superalgebra $s\mathcal{M}$ and the N -extended Maxwell superalgebra $s\mathcal{M}^{(N)}$ recently found by the Maurer–Cartan expansion procedure, are derived alternatively as an S -expansion of $\mathfrak{osp}(4|N)$. Moreover, we show that new minimal Maxwell superalgebras type $s\mathcal{M}_{m+2}$ and their N -extended generalization can be obtained using the S -expansion procedure.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/3.0/>). Funded by SCOAP³.

1. Introduction

The derivation of new Lie algebras from a given one is particularly interesting in Physics since it allows us to find new physical theories from an already known. In fact, an important example consists in obtaining the Poincaré algebra from the Galileo algebra using a deformation process which can be seen as an algebraic prediction of Relativity. At the present, there are at least four different ways to relate new Lie algebras. In particular, the expansion method leads to higher-dimensional new Lie algebra from a given one. The expansion procedure was first

<http://dx.doi.org/10.1016/j.nuclphysb.2014.07.022>

0550-3213/© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/3.0/>). Funded by SCOAP³.

introduced by Hadsuda and Sakaguchi in Ref. [1] in the context of Ads superstring. An interesting expansion method was proposed by Azcarraga, Izquierdo, Picón and Varela in Ref. [2] and subsequently developed in Refs. [3,4]. This expansion method known as Maurer–Cartan (MC) forms power-series expansion consists in rescaling some group parameters by a factor λ , and then applying an expansion as a power series in λ . This series is truncated in a way that the Maurer–Cartan equations of the new algebra are satisfied.

Another expansion method was proposed by Izaurieta, Rodríguez and Salgado in Ref. [5] which is based on operations performed directly on the algebra generators. This method consists in combining the inner multiplication law of a semigroup S with the structure constants of a Lie algebra \mathfrak{g} in order to define the Lie bracket of a new algebra $\mathfrak{G} = S \times \mathfrak{g}$. This Abelian Semigroup expansion procedure can reproduce all Maurer–Cartan forms of power series expansion for a particular choice of a semigroup S . Interestingly, different choices of the semigroup yield to new expanded Lie algebras that cannot be obtained by the MC expansion.

An important property of the S -expansion procedure is that it provides us with an invariant tensor for the S -expanded algebra in terms of an invariant tensor for the original algebra. This is particularly useful in order to construct Chern–Simons and Born–Infeld like actions.

Some examples of algebras obtained as an S -expansion can be found in Refs. [5,6] where the D’Auria–Fré superalgebra introduced originally in Ref. [7] and the M algebra are derived alternatively as an S -expansion of $\mathfrak{osp}(32|1)$. Subsequently, in Refs. [8,9] it was shown that standard odd-dimensional General Relativity can be obtained from Chern–Simons gravity theory for a certain Lie algebra \mathfrak{B}_m and recently it was found that standard even-dimensional General Relativity emerges as a limit of a Born–Infeld like theory invariant under a certain subalgebra of the Lie algebra \mathfrak{B}_m [9,11]. Very recently it was found that the so-called \mathfrak{B}_m Lie algebra corresponds to the Maxwell algebras type¹ \mathcal{M}_m [10]. These Maxwell algebras type \mathcal{M}_m can be obtained as an S -expansion for the Ads algebra using $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$ as an abelian semigroup.

The Maxwell algebra has been extensively studied in Refs. [12–22]. This algebra describes the symmetries of a particle moving in a background in the presence of a constant electromagnetic field [12]. Interestingly, in Ref. [17] it was shown that a Maxwell extension of Einstein gravity leads to a generalized cosmological term. Furthermore, it was introduced in Ref. [15] the minimal $D = 4$ Maxwell superalgebra $s\mathcal{M}$ which contains the Maxwell algebra as its bosonic subalgebra. In Ref. [19] the Maurer–Cartan expansion was used in order to obtain the minimal Maxwell superalgebra and its N -extended generalization from the $\mathfrak{osp}(4|N)$ superalgebra. This Maxwell superalgebra may be used to obtain the minimal $D = 4$ pure supergravity from the 2-form curvature associated to $s\mathcal{M}$ [21].

The purpose of this work is to show that the abelian semigroup expansion is an alternative expansion method to obtain the Maxwell superalgebra and the N -extended cases. In this way we show that the results of Ref. [19] can be derived alternatively as an S -expansion of the $\mathfrak{osp}(4|N)$ superalgebra choosing appropriate semigroups. In particular, the minimal Maxwell superalgebra $s\mathcal{M}$ is obtained as an S -expansion setting a generator equals to zero. We finally generalize these results proposing new Maxwell superalgebras namely, the minimal Maxwell superalgebras type $s\mathcal{M}_{m+2}$ and the N -extended superalgebras $s\mathcal{M}_{m+2}^{(N)}$ which can be derived from the $\mathfrak{osp}(4|N)$ superalgebra using the S -expansion procedure.

The new $s\mathcal{M}_{m+2}$ superalgebras introduced here are obtained by applying the S -expansion to the $\mathfrak{osp}(4|1)$ superalgebra and can be seen as supersymmetric extensions of the Maxwell alge-

¹ Alternatively known as generalized Poincaré algebra.

bras type introduced in Ref. [9]. Unlike the Maxwell superalgebra $s\mathcal{M}$ these new superalgebras involve a larger number of extra fermionic generators depending on the value of m . Furthermore, these superalgebras could be used to construct dynamical actions in $D = 4$ leading to standard supergravity, in a very similar way to the bosonic case considered in Ref. [9].

This work is organized as follows: In Section 2, we briefly review some aspects of the S -expansion procedure which will be helpful to understand this work. In Section 3, we review an interesting application of S -expansion procedure in order to get the Maxwell algebra family. Sections 4 and 5 contain our main results. In Section 4, we present the different minimal $D = 4$ Maxwell superalgebras which can be obtained from an S -expansion of $\mathfrak{osp}(4|1)$ superalgebra. In Section 5, we extend these results to the N -extended case and we present different N -extended $D = 4$ Maxwell superalgebras from an S -expansion of the $\mathfrak{osp}(4|N)$ superalgebra. Section 6 concludes the work with a comment about possible developments.

2. The S -expansion procedure

In this section, we shall review the main aspects of the abelian semigroup expansion method introduced in Ref. [5]. Let us consider a Lie (super)algebra \mathfrak{g} with basis T_A and a finite abelian semigroup $S = \{\lambda_\alpha\}$. Then, the direct product $\mathfrak{G} = S \times \mathfrak{g}$ is also a Lie algebra given by

$$[T_{(A,\alpha)}, T_{(B,\beta)}] = K_{\alpha\beta}{}^\gamma C_{AB}{}^C T_{(C,\gamma)}. \quad (1)$$

The S -expansion procedure consist in combining the inner multiplication law of a semigroup S with the structure constants of a Lie algebra \mathfrak{g} . Interestingly, there are different ways of extracting smaller algebras from $\mathfrak{G} = S \times \mathfrak{g}$. But before to extract smaller algebras it is necessary to decompose the original algebra \mathfrak{g} into a direct sum of subspaces $\mathfrak{g} = \bigoplus_{p \in I} V_p$, where I is a set of indices. Then for each $p, q \in I$ it is possible to define $i_{(p,q)} \subset I$ such that

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r. \quad (2)$$

Following the definitions of Ref. [5], it is possible to define a subset decomposition $S = \bigcup_{p \in I} S_p$ of the semigroup S such that

$$S_p \cdot S_q \subset \bigcap_{r \in i_{(p,q)}} S_r. \quad (3)$$

When such subset decomposition exists, then we say that

$$\mathfrak{G}_R = \bigoplus_{p \in I} S_p \times V_p, \quad (4)$$

is a resonant subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

Another case of smaller algebra is when there is a zero element in the semigroup $0_S \in S$, such that for all $\lambda_\alpha \in S$, we have $0_S \lambda_\alpha = 0_S$. In this case, it is possible to reduce the original algebra imposing $0_S \times T_A = 0$ and obtain a new Lie (super)algebra.

Interestingly, there is a way to extract a reduced algebra from a resonant subalgebra. Let $\mathfrak{G}_R = \bigoplus_p S_p \times V_p$ be a resonant subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ such that

$$\hat{S}_p \cap \check{S}_p = \emptyset, \quad (5)$$

$$\check{S}_p \cdot \hat{S}_q \subset \bigcap_{r \in i(p,q)} \hat{S}_r. \quad (6)$$

Then, these conditions induce the decomposition

$$\check{\mathfrak{G}}_R = \bigoplus_{p \in I} \check{S}_p \times V_p, \quad (7)$$

$$\hat{\mathfrak{G}}_R = \bigoplus_{p \in I} \hat{S}_p \times V_p, \quad (8)$$

with

$$[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R] \subset \hat{\mathfrak{G}}_R, \quad (9)$$

and therefore $[\check{\mathfrak{G}}_R]$ corresponds to a reduced algebra of \mathfrak{G}_R . The proofs of these definitions can be found in Ref. [5].

We can see that if we want to obtain an S -expanded algebra, we only need to solve the resonance condition for an abelian semigroup S . In the next section we will briefly review an interesting application of S -expansion procedure in order to derive the Maxwell algebra family. Then we shall see how it is possible to extend this result to the supersymmetric case.

3. Maxwell algebra as an S -expansion

In order to describe how the S -expansion procedure works, let us review here the results obtained in Refs. [9,10]. The symmetries of a particle moving in a background in presence of a constant electromagnetic field are described by the Maxwell algebra \mathcal{M} . This algebra is provided by $\{J_{ab}, P_a, Z_{ab}\}$ where $\{P_a, J_{ab}\}$ do not generate the Poincaré algebra. In fact a particular characteristic of the Maxwell algebra is given by the relation

$$[P_a, P_b] = Z_{ab} \quad (10)$$

where Z_{ab} commutes with all generators of the algebra except the Lorentz generators J_{ab} ,

$$[J_{ab}, Z_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (11)$$

$$[Z_{ab}, P_a] = [Z_{ab}, Z_{cd}] = 0. \quad (12)$$

The other commutators of the algebra are

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (13)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b. \quad (14)$$

Following Refs. [9,10], it is possible to obtain the Maxwell algebra \mathcal{M} as an S -expansion of the Ads Lie algebra \mathfrak{g} using $S_E^{(2)}$ as the abelian semigroup.

Before to apply the S -expansion procedure it is necessary to consider a decomposition of the original algebra \mathfrak{g} in subspaces V_p ,

$$\mathfrak{g} = \mathfrak{so}(3, 2) = \mathfrak{so}(3, 1) \oplus \frac{\mathfrak{so}(3, 2)}{\mathfrak{so}(3, 1)} = V_0 \oplus V_1, \quad (15)$$

where V_0 is generated by the Lorentz generator \tilde{J}_{ab} and V_1 is generated by the Ads boost generator \tilde{P}_a . The \tilde{J}_{ab} , \tilde{P}_a generators satisfy the following relations

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{bc} \tilde{J}_{ad} - \eta_{ac} \tilde{J}_{bd} - \eta_{bd} \tilde{J}_{ac} + \eta_{ad} \tilde{J}_{bc}, \quad (16)$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{bc} \tilde{P}_a - \eta_{ac} \tilde{P}_b, \quad (17)$$

$$[\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab}. \quad (18)$$

The subspace structure may be written as

$$[V_0, V_0] \subset V_0, \quad (19)$$

$$[V_0, V_1] \subset V_1, \quad (20)$$

$$[V_1, V_1] \subset V_0. \quad (21)$$

Let $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ be an abelian semigroup with the following subset decomposition $S_E^{(2)} = S_0 \cup S_1$, where the subsets S_0, S_1 are given by

$$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}, \quad (22)$$

$$S_1 = \{\lambda_1, \lambda_3\}, \quad (23)$$

where λ_3 corresponds to the zero element of the semigroup ($0_s = \lambda_3$). This subset decomposition is said to be “resonant” because it satisfies [compare with Eqs. (19)–(21)].

$$S_0 \cdot S_0 \subset S_0, \quad (24)$$

$$S_0 \cdot S_1 \subset S_1, \quad (25)$$

$$S_1 \cdot S_1 \subset S_0. \quad (26)$$

In this case, the elements of the semigroup $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ satisfy the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 3, \\ \lambda_3, & \text{when } \alpha + \beta > 3. \end{cases} \quad (27)$$

Following the definitions of Ref. [5], after extracting a resonant subalgebra and performing its 0_S -reduction, one finds the Maxwell algebra $\mathcal{M} = \{J_{ab}, P_a, Z_{ab}\}$, whose generators can be written in terms of the original ones,

$$J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}, \quad (28)$$

$$P_a = \lambda_1 \otimes \tilde{P}_a, \quad (29)$$

$$Z_{ab} = \lambda_2 \otimes \tilde{J}_{ab}. \quad (30)$$

Interestingly, in Refs. [9,11], it was shown that standard four-dimensional General Relativity emerges as a limit of a Born–Infeld theory invariant under a certain subalgebra of the Maxwell algebra \mathcal{M} , which was denoted by² $\mathcal{L}^{\mathcal{M}}$. It was shown that this subalgebra can be obtained as an S -expansion of the Lorentz algebra $\mathfrak{so}(3, 1)$.

It is possible to extend this procedure and obtain all the possible Maxwell algebras type using the appropriate semigroup.

Following Ref. [9], let us consider the S -expansion of the Lie algebra $\mathfrak{so}(3, 2)$ using the abelian semigroup $S_E^{(2n-1)} = \{\lambda_0, \lambda_1, \dots, \lambda_{2n}\}$ with the following multiplication law

² Initially called as $\mathcal{L}^{\mathfrak{B}_4}$ algebra.

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 2n, \\ \lambda_{2n}, & \text{when } \alpha + \beta > 2n. \end{cases} \quad (31)$$

The λ_α elements are dimensionless and can be represented by the set of $2n \times 2n$ matrices $[\lambda_\alpha]^i_j = \delta^i_{j+\alpha}$, where $i, j = 1, \dots, 2n-1$; $\alpha = 0, \dots, 2n$, and δ stands for the Kronecker delta.

After extracting a resonant subalgebra and performing its $0_s (= \lambda_{2n})$ -reduction, one finds the Maxwell algebra type³ \mathcal{M}_{2n+1} , whose generators are related to the original ones,

$$J_{ab} = J_{(ab,0)} = \lambda_0 \otimes \tilde{J}_{ab}, \quad (32)$$

$$P_a = P_{(a,1)} = \lambda_1 \otimes \tilde{P}_a, \quad (33)$$

$$Z_{ab}^{(i)} = J_{(ab,2i)} = \lambda_{2i} \otimes \tilde{J}_{ab}, \quad (34)$$

$$Z_a^{(i)} = P_{(a,2i+1)} = \lambda_{2i+1} \otimes \tilde{P}_a, \quad (35)$$

with $i = 0, \dots, n-1$. The commutators of the algebra are

$$[P_a, P_b] = Z_{ab}^{(1)}, \quad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad (36)$$

$$[J_{ab}, J_{cd}] = \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb}, \quad (37)$$

$$[J_{ab}, Z_c^{(i)}] = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \quad (38)$$

$$[Z_{ab}^{(i)}, P_c] = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \quad (39)$$

$$[Z_{ab}^{(i)}, Z_c^{(j)}] = \eta_{bc} Z_a^{(i+j)} - \eta_{ac} Z_b^{(i+j)}, \quad (40)$$

$$[J_{ab}, Z_{cd}^{(i)}] = \eta_{cb} Z_{ad}^{(i)} - \eta_{ca} Z_{bd}^{(i)} + \eta_{db} Z_{ca}^{(i)} - \eta_{da} Z_{cb}^{(i)}, \quad (41)$$

$$[Z_{ab}^{(i)}, Z_{cd}^{(j)}] = \eta_{cb} Z_{ad}^{(i+j)} - \eta_{ca} Z_{bd}^{(i+j)} + \eta_{db} Z_{ca}^{(i+j)} - \eta_{da} Z_{cb}^{(i+j)}, \quad (42)$$

$$[P_a, Z_c^{(i)}] = Z_{ab}^{(i+1)}, \quad [Z_a^{(i)}, Z_c^{(j)}] = Z_{ab}^{(i+j+1)}. \quad (43)$$

We note that setting $Z_{ab}^{(i+1)}$ and $Z_a^{(i)}$ equal to zero, we reobtain the Maxwell algebra \mathcal{M} . In fact, every Maxwell algebra type \mathcal{M}_l can be obtained from \mathcal{M}_{2m+1} setting some bosonic generators equal to zero. These algebras are particularly interesting in gravity context, since it was shown in [9] that standard odd-dimensional general relativity may emerge as the weak coupling constant limit of $(2p+1)$ -dimensional Chern–Simons Lagrangian invariant under the Maxwell algebra type \mathcal{M}_{2m+1} , if and only if $m \geq p$. Similarly, it was shown that standard even-dimensional general relativity emerges as the weak coupling constant limit of a $(2p)$ -dimensional Born–Infeld type Lagrangian invariant under a subalgebra $\mathcal{L}^{\mathcal{M}_{2m}}$ of the Maxwell algebra type, if and only if $m \geq p$.

4. S-expansion of the $\mathfrak{osp}(4|1)$ superalgebra

In this section, we shall take the Ads superalgebra $\mathfrak{osp}(4|1)$ as a starting point. We will see that different choices of abelian semigroup S lead to new interesting $D = 4$ superalgebras. In every case, before to apply the S -expansion procedure it is necessary to decompose the original algebra \mathfrak{g} as a direct sum of subspaces V_p ,

³ Initially called as \mathfrak{B}_{2n+1} algebra.

$$\begin{aligned} \mathfrak{g} = \mathfrak{osp}(4|1) &= \mathfrak{so}(3, 1) \oplus \frac{\mathfrak{osp}(4|1)}{\mathfrak{sp}(4)} \oplus \frac{\mathfrak{sp}(4)}{\mathfrak{so}(3, 1)} \\ &= V_0 \oplus V_1 \oplus V_2, \end{aligned} \quad (44)$$

where V_0 corresponds to the Lorentz subspace generated by \tilde{J}_{ab} , V_1 corresponds to the fermionic subspace generated by a 4-component Majorana spinor charge \tilde{Q}_α and V_2 corresponds to the Ads boost generated by \tilde{P}_a . The $\mathfrak{osp}(4|1)$ (anti)commutation relations read

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \quad (45)$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \quad (46)$$

$$[\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab}, \quad (47)$$

$$[\tilde{J}_{ab}, \tilde{Q}_\alpha] = -\frac{1}{2}(\gamma_{ab}\tilde{Q})_\alpha, \quad [\tilde{P}_a, \tilde{Q}_\alpha] = -\frac{1}{2}(\gamma_a\tilde{Q})_\alpha, \quad (48)$$

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = -\frac{1}{2}[(\gamma^{ab}C)_{\alpha\beta}\tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta}\tilde{P}_a]. \quad (49)$$

Here, C stands for the charge conjugation matrix and γ_a are Dirac matrices.

The subspace structure may be written as

$$[V_0, V_0] \subset V_0, \quad (50)$$

$$[V_0, V_1] \subset V_1, \quad (51)$$

$$[V_0, V_2] \subset V_2, \quad (52)$$

$$[V_1, V_1] \subset V_0 \oplus V_2, \quad (53)$$

$$[V_1, V_2] \subset V_1, \quad (54)$$

$$[V_2, V_2] \subset V_0. \quad (55)$$

The next step consists of finding a subset decomposition of a semigroup S which is “resonant” with respect to (50)–(55).

4.1. Minimal $D = 4$ super-Maxwell algebra

Let us consider $S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ as the relevant finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 5, \\ \lambda_5, & \text{when } \alpha + \beta > 5. \end{cases} \quad (56)$$

In this case, λ_5 plays the role of the zero element of the semigroup $S_E^{(4)}$, so we have for each $\lambda_\alpha \in S_E^{(4)}$, $\lambda_5 \lambda_\alpha = \lambda_5 = 0_s$. Let us consider the decomposition $S = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}, \quad (57)$$

$$S_1 = \{\lambda_1, \lambda_3, \lambda_5\}, \quad (58)$$

$$S_2 = \{\lambda_2, \lambda_4, \lambda_5\}. \quad (59)$$

One sees that this decomposition is resonant since it satisfies the same structure as the subspaces V_p [compare with Eqs. (50)–(55)]

$$S_0 \cdot S_0 \subset S_0, \quad (60)$$

$$S_0 \cdot S_1 \subset S_1, \quad (61)$$

$$S_0 \cdot S_2 \subset S_2, \quad (62)$$

$$S_1 \cdot S_1 \subset S_0 \cap S_2, \quad (63)$$

$$S_1 \cdot S_2 \subset S_1, \quad (64)$$

$$S_2 \cdot S_2 \subset S_0. \quad (65)$$

Following theorem IV.2 of Ref. [5], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \quad (66)$$

is a resonant super-subalgebra of $S_E^{(4)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \times \{\tilde{J}_{ab}\} = \{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_4 \tilde{J}_{ab}, \lambda_5 \tilde{J}_{ab}\}, \quad (67)$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3, \lambda_5\} \times \{\tilde{Q}_\alpha\} = \{\lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\alpha, \lambda_5 \tilde{Q}_\alpha\}, \quad (68)$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2, \lambda_4, \lambda_5\} \times \{\tilde{P}_a\} = \{\lambda_2 \tilde{P}_a, \lambda_4 \tilde{P}_a, \lambda_5 \tilde{P}_a\}. \quad (69)$$

In order to extract a smaller superalgebra from the resonant super-subalgebra \mathfrak{G}_R it is necessary to apply the reduction procedure.

Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_0, \lambda_2, \lambda_4\}, \quad \hat{S}_0 = \{\lambda_5\}, \quad (70)$$

$$\check{S}_1 = \{\lambda_1, \lambda_3\}, \quad \hat{S}_1 = \{\lambda_5\}, \quad (71)$$

$$\check{S}_2 = \{\lambda_2\}, \quad \hat{S}_2 = \{\lambda_4, \lambda_5\}. \quad (72)$$

For each p , $\hat{S}_p \cap \check{S}_p = \emptyset$, and using the product (56) one sees that the partition satisfies [compare with Eqs. (50)–(55)]

$$\begin{aligned} \check{S}_0 \cdot \hat{S}_0 &\subset \hat{S}_0, & \check{S}_1 \cdot \hat{S}_1 &\subset \hat{S}_0 \cap \hat{S}_2, \\ \check{S}_0 \cdot \hat{S}_1 &\subset \hat{S}_1, & \check{S}_1 \cdot \hat{S}_2 &\subset \hat{S}_1, \\ \check{S}_0 \cdot \hat{S}_2 &\subset \hat{S}_2, & \check{S}_2 \cdot \hat{S}_2 &\subset \hat{S}_0. \end{aligned} \quad (73)$$

Then, following definitions of Ref. [5], we have

$$\check{\mathfrak{G}}_R = (\check{S}_0 \times V_0) \oplus (\check{S}_1 \times V_1) \oplus (\check{S}_2 \times V_2), \quad (74)$$

$$\hat{\mathfrak{G}}_R = (\hat{S}_0 \times V_0) \oplus (\hat{S}_1 \times V_1) \oplus (\hat{S}_2 \times V_2), \quad (75)$$

where

$$[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R] \subset \hat{\mathfrak{G}}_R, \quad (76)$$

and therefore $|\check{\mathfrak{G}}_R|$ corresponds to a reduced algebra of \mathfrak{G}_R . These S -expansion process can be seen explicitly in the following diagrams:

λ_5	$J_{ab,5}$	$Q_{\alpha,5}$	$P_{a,5}$	λ_5				(77)
λ_4	$J_{ab,4}$		$P_{a,4}$	λ_4	$J_{ab,4}$			
λ_3		$Q_{\alpha,3}$		λ_3		$Q_{\alpha,3}$		
λ_2	$J_{ab,2}$		$P_{a,2}$	λ_2	$J_{ab,2}$		$P_{a,2}$	
λ_1		$Q_{\alpha,1}$		λ_1		$Q_{\alpha,1}$		
λ_0	$J_{ab,0}$			λ_0	$J_{ab,0}$			
	V_0	V_1	V_2		V_0	V_1	V_2	

where we have defined $J_{ab,i} = \lambda_i \tilde{J}_{ab}$, $P_{a,i} = \lambda_i \tilde{P}_a$ and $Q_{\alpha,i} = \lambda_i \tilde{Q}_\alpha$. We can observe that the first diagram corresponds to the resonant subalgebra of the S -expanded superalgebra $S_E^{(4)} \times \mathfrak{osp}(4|1)$. The second one consists in a particular reduction of the resonant subalgebra.

Thus, the new superalgebra obtained is generated by $\{J_{ab}, P_a, \tilde{Z}_{ab}, Z_{ab}, Q_\alpha, \Sigma_\alpha\}$ where these new generators can be written as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, \\ \tilde{Z}_{ab} &= J_{ab,2} = \lambda_2 \tilde{J}_{ab}, & Z_{ab} &= J_{ab,4} = \lambda_4 \tilde{J}_{ab}, \\ Q_\alpha &= Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, & \Sigma_\alpha &= Q_{\alpha,3} = \lambda_3 \tilde{Q}_\alpha. \end{aligned} \quad (78)$$

These new generators satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (79)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad [P_a, P_b] = Z_{ab}, \quad (80)$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (81)$$

$$[P_a, Q_\alpha] = -\frac{1}{2}(\gamma_a \Sigma)_\alpha, \quad (82)$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} Q)_\alpha, \quad (83)$$

$$[J_{ab}, \Sigma_\alpha] = -\frac{1}{2}(\gamma_{ab} \Sigma)_\alpha, \quad (84)$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab} - 2(\gamma^a C)_{\alpha\beta} P_a], \quad (85)$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2}(\gamma^{ab} C)_{\alpha\beta} Z_{ab}, \quad (86)$$

$$[J_{ab}, \tilde{Z}_{ab}] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \quad (87)$$

$$[\tilde{Z}_{ab}, \tilde{Z}_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (88)$$

$$[\tilde{Z}_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} \Sigma)_\alpha, \quad (89)$$

$$\text{others} = 0, \quad (90)$$

where we have used the multiplication law of the semigroup (56) and the commutation relations of the original superalgebra (see Appendix A). The new superalgebra obtained after a reduced resonant S -expansion of $\mathfrak{osp}(4|1)$ superalgebra corresponds to a generalized minimal super-Maxwell algebra $s\mathcal{M}_4$ in $D = 4$. One can see that imposing $\tilde{Z}_{ab} = 0$ leads us to the minimal super-Maxwell algebra $s\mathcal{M}$ [17,19]. This can be done since the Jacobi identities for spinors generators are satisfied due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ).

In this case, the S -expansion procedure produces a new Majorana spinor charge Σ . The introduction of a second abelian spinorial generator has been initially proposed in Ref. [7] in the context of $D = 11$ supergravity and subsequently in Ref. [22] in the context of superstring theory.

The $s\mathcal{M}$ superalgebra seems particularly interesting in the context of $D = 4$ supergravity. In fact in Ref. [21], it was shown that $D = 4$, $N = 1$ pure supergravity Lagrangian can be written as a quadratic expression in the curvatures of the gauge fields associated with the minimal super-Maxwell algebra.

It is interesting to note that the expanded superalgebra contains the Maxwell algebra $\mathcal{M} = \{J_{ab}, P_a, Z_{ab}\}$ and the Lorentz type subalgebra $\mathcal{L}^{\mathcal{M}} = \{J_{ab}, Z_{ab}\}$ introduced in Ref. [11] as subalgebras.

4.2. Minimal $D = 4$ super-Maxwell algebra type $s\mathcal{M}_5$

In Ref. [9], it was shown that the Maxwell algebra type \mathcal{M}_m can be obtained from an S -expansion of Ads algebra. These bigger algebras require semigroups with more elements but with the same type of multiplication law. Since our main motivation is to obtain a $D = 4$ super-Maxwell algebra type $s\mathcal{M}_m$ it seems natural to consider a semigroup bigger than $S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. As in the previous case, we shall consider $\mathfrak{g} = \mathfrak{osp}(4|1)$ as a starting point with the subspace structure given by Eqs. (50)–(55).

Let us consider $S_E^{(6)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ as the relevant finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 7, \\ \lambda_7, & \text{when } \alpha + \beta > 7, \end{cases} \quad (91)$$

where λ_7 plays the role of the zero element of the semigroup $S_E^{(6)}$. Let us consider the decomposition $S = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_6, \lambda_7\}, \quad (92)$$

$$S_1 = \{\lambda_1, \lambda_3, \lambda_5, \lambda_7\}, \quad (93)$$

$$S_2 = \{\lambda_2, \lambda_4, \lambda_6, \lambda_7\}. \quad (94)$$

This subset decomposition of $S_E^{(6)}$ satisfies the resonance condition since it satisfies the same structure that the subspaces V_p [compare with Eqs. (50)–(55)]

$$S_0 \cdot S_0 \subset S_0, \quad S_1 \cdot S_1 \subset S_0 \cap S_2, \quad (95)$$

$$S_0 \cdot S_1 \subset S_1, \quad S_1 \cdot S_2 \subset S_1, \quad (96)$$

$$S_0 \cdot S_2 \subset S_2, \quad S_2 \cdot S_2 \subset S_0. \quad (97)$$

Therefore, according to Theorem IV.2 of Ref. [5], we have that

$$\mathfrak{G}_R = W_0 + W_1 + W_2, \quad (98)$$

with

$$W_p = S_p \times V_p, \quad (99)$$

is a resonant super-subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

As in the previous case, it is possible to extract a smaller superalgebra from the resonant super-subalgebra \mathfrak{G}_R using the reduction procedure. Let $S_p = \check{S}_p \cup \hat{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_0, \lambda_2, \lambda_4\}, \quad \hat{S}_0 = \{\lambda_6, \lambda_7\}, \quad (100)$$

$$\check{S}_1 = \{\lambda_1, \lambda_3, \lambda_5\}, \quad \hat{S}_1 = \{\lambda_7\}, \quad (101)$$

$$\check{S}_2 = \{\lambda_2, \lambda_4, \lambda_6\}, \quad \hat{S}_2 = \{\lambda_7\}. \quad (102)$$

For each p , $\hat{S}_p \cap \check{S}_p = \emptyset$, and using the product (91) one sees that the partition satisfies [compare with Eqs. (50)–(55)]

$$\begin{aligned} \check{S}_0 \cdot \hat{S}_0 &\subset \hat{S}_0, & \check{S}_1 \cdot \hat{S}_1 &\subset \hat{S}_0 \cap \hat{S}_2, \\ \check{S}_0 \cdot \hat{S}_1 &\subset \hat{S}_1, & \check{S}_1 \cdot \hat{S}_2 &\subset \hat{S}_1, \\ \check{S}_0 \cdot \hat{S}_2 &\subset \hat{S}_2, & \check{S}_2 \cdot \hat{S}_2 &\subset \hat{S}_0. \end{aligned} \quad (103)$$

Then, we have

$$\check{\mathfrak{G}}_R = (\check{S}_0 \times V_0) \oplus (\check{S}_1 \times V_1) \oplus (\check{S}_2 \times V_2), \quad (104)$$

$$\hat{\mathfrak{G}}_R = (\hat{S}_0 \times V_0) \oplus (\hat{S}_1 \times V_1) \oplus (\hat{S}_2 \times V_2), \quad (105)$$

where

$$[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R] \subset \hat{\mathfrak{G}}_R, \quad (106)$$

and therefore $|\check{\mathfrak{G}}_R|$ corresponds to a reduced algebra of $\hat{\mathfrak{G}}_R$. These procedures can be saw explicitly in the following diagrams:

$$\begin{array}{c} \lambda_7 \\ \lambda_6 \\ \lambda_5 \\ \lambda_4 \\ \lambda_3 \\ \lambda_2 \\ \lambda_1 \\ \lambda_0 \end{array} \begin{array}{|c|c|c|} \hline J_{ab,7} & Q_{\alpha,7} & P_{a,7} \\ \hline J_{ab,6} & & P_{a,6} \\ \hline & Q_{\alpha,5} & \\ \hline J_{ab,4} & & P_{a,4} \\ \hline & Q_{\alpha,3} & \\ \hline J_{ab,2} & & P_{a,2} \\ \hline & Q_{\alpha,1} & \\ \hline J_{ab,0} & & \\ \hline \end{array} \begin{array}{c} \lambda_7 \\ \lambda_6 \\ \lambda_5 \\ \lambda_4 \\ \lambda_3 \\ \lambda_2 \\ \lambda_1 \\ \lambda_0 \end{array} \begin{array}{|c|c|c|} \hline & & \\ \hline & & P_{a,6} \\ \hline & Q_{\alpha,5} & \\ \hline J_{ab,4} & & P_{a,4} \\ \hline & Q_{\alpha,3} & \\ \hline J_{ab,2} & & P_{a,2} \\ \hline & Q_{\alpha,1} & \\ \hline J_{ab,0} & & \\ \hline \end{array} \begin{array}{c} V_0 \\ V_1 \\ V_2 \end{array} \quad (107)$$

where we have defined $J_{ab,i} = \lambda_i \tilde{J}_{ab}$, $P_{a,i} = \lambda_i \tilde{P}_a$ and $Q_{\alpha,i} = \lambda_i \tilde{Q}_\alpha$. The first diagram corresponds to the resonant subalgebra of the S -expanded superalgebra $S_E^{(6)} \times \mathfrak{osp}(4|1)$. The second one consists in a particular reduction of the resonant subalgebra.

The new superalgebra is generated by $\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, Z_a, \tilde{Z}_a, Q_\alpha, \Sigma_\alpha, \Phi_\alpha\}$ where these new generators can be written as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & \tilde{Z}_a &= P_{a,4} = \lambda_4 \tilde{P}_a, \\ P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, & Q_\alpha &= Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, \\ Z_{ab} &= J_{ab,4} = \lambda_4 \tilde{J}_{ab}, & \Sigma_\alpha &= Q_{\alpha,3} = \lambda_3 \tilde{Q}_\alpha, \\ \tilde{Z}_{ab} &= J_{ab,2} = \lambda_2 \tilde{J}_{ab}, & \Phi_\alpha &= Q_{\alpha,5} = \lambda_5 \tilde{Q}_\alpha, \\ Z_a &= P_{a,6} = \lambda_6 \tilde{P}_a. \end{aligned} \quad (108)$$

These new generators satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (109)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad [P_a, P_b] = Z_{ab}, \quad (110)$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (111)$$

$$[Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \quad [J_{ab}, Z_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \quad (112)$$

$$[\tilde{Z}_{ab}, \tilde{Z}_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (113)$$

$$[J_{ab}, \tilde{Z}_{cd}] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \quad (114)$$

$$[\tilde{Z}_{ab}, P_c] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \quad [\tilde{Z}_{ab}, \tilde{Z}_c] = \eta_{bc} Z_a - \eta_{ac} Z_b \quad (115)$$

$$[J_{ab}, \tilde{Z}_c] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \quad (116)$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} Q)_\alpha, \quad [J_{ab}, \Sigma_\alpha] = -\frac{1}{2}(\gamma_{ab} \Sigma)_\alpha, \quad (117)$$

$$[J_{ab}, \Phi_\alpha] = -\frac{1}{2}(\gamma_{ab} \Phi)_\alpha, \quad [\tilde{Z}_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} \Sigma)_\alpha, \quad (118)$$

$$[\tilde{Z}_{ab}, \Sigma_\alpha] = -\frac{1}{2}(\gamma_{ab} \Phi)_\alpha, \quad [Z_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} \Phi)_\alpha, \quad (119)$$

$$[P_a, Q_\alpha] = -\frac{1}{2}(\gamma_a \Sigma)_\alpha, \quad [P_a, \Sigma_\alpha] = -\frac{1}{2}(\gamma_a \Phi)_\alpha, \quad (120)$$

$$[\tilde{Z}_a, Q_\alpha] = -\frac{1}{2}(\gamma_a \Phi)_\alpha, \quad (121)$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab} - 2(\gamma^a C)_{\alpha\beta} P_a], \quad (122)$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} Z_{ab} - 2(\gamma^a C)_{\alpha\beta} \tilde{Z}_a], \quad (123)$$

$$\{Q_\alpha, \Phi_\beta\} = (\gamma^a C)_{\alpha\beta} Z_a = \{\Sigma_\alpha, \Sigma_\beta\}, \quad (124)$$

$$\text{others} = 0, \quad (125)$$

where we have used the multiplication law of the semigroup (91) and the commutation relations of the original superalgebra (45)–(49). The new superalgebra obtained after a reduced resonant S -expansion of $\mathfrak{osp}(4|1)$ superalgebra corresponds to a minimal Maxwell superalgebra type $s\mathcal{M}_5$ in $D = 4$. Interestingly, this new superalgebra contains the Maxwell algebra type $\mathcal{M}_5 = \{J_{ab}, P_a, Z_{ab}, Z_a\}$ as a subalgebra [8,9].

In this case, the S -expansion method produces two new Majorana spinors charge Σ and Φ . These fermionic generators transform as spinors under Lorentz transformations. One sees that the minimal super-Maxwell type $s\mathcal{M}_5$ requires new bosonic generators $(\tilde{Z}_{ab}, \tilde{Z}_a, Z_a)$ and Σ is not abelian anymore. It is important to note that setting \tilde{Z}_{ab} and \tilde{Z}_a equal to zero does not lead to a subalgebra. In fact, these generators are required in Jacobi identity for $(Q_\alpha, Q_\beta, \Sigma_\gamma)$ due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = (C\gamma^{a\beta})_{(\alpha\beta}(C\gamma_{a\beta})_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ).

It would be interesting to study this algebraic structure in the context of supergravity theory. It seems that the new minimal Maxwell superalgebra $s\mathcal{M}_5$ defined here may enlarge the $D = 4$ pure supergravity Lagrangian in a particular way.

4.3. Minimal $D = 4$ super-Maxwell algebra type $s\mathcal{M}_{m+2}$

Let us further generalize the previous setting. In order to obtain the minimal $D = 4$ super-Maxwell algebra type $s\mathcal{M}_{m+2}$, it is necessary to consider a bigger semigroup. Let us consider $S_E^{(2m)} = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{2m+1}\}$ as the relevant finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq \lambda_{2m+1}, \\ \lambda_{2m+1}, & \text{when } \alpha + \beta > \lambda_{2m+1}. \end{cases} \quad (126)$$

where λ_{2m+1} plays the role of the zero element of the semigroup. Let us consider the decomposition $S_E^{(2m)} = S_0 \cup S_1 \cup S_2$, where the subsets S_0, S_1, S_2 are given by the general expression

$$S_p = \left\{ \lambda_{2n+p}, \text{ with } n = 0, \dots, \left\lfloor \frac{2m-p}{2} \right\rfloor \right\} \cup \{\lambda_{2m+1}\}, \quad p = 0, 1, 2. \quad (127)$$

This decomposition is said to be resonant since it satisfies [compare with Eqs. (50)–(55)]

$$S_0 \cdot S_0 \subset S_0, \quad S_1 \cdot S_1 \subset S_0 \cap S_2, \quad (128)$$

$$S_0 \cdot S_1 \subset S_1, \quad S_1 \cdot S_2 \subset S_1, \quad (129)$$

$$S_0 \cdot S_2 \subset S_2, \quad S_2 \cdot S_2 \subset S_0. \quad (130)$$

Therefore, we have

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \quad (131)$$

with

$$W_p = S_p \times V_p, \quad (132)$$

is a resonant subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

As in previous cases, it is possible to extract a smaller algebra from the resonant subalgebra \mathfrak{G}_R using the reduction procedure. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_{2n}, \text{ with } n = 0, \dots, 2[m/2]\}, \quad \hat{S}_0 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \quad (133)$$

$$\check{S}_1 = \{\lambda_{2n+1}, \text{ with } n = 0, \dots, m-1\}, \quad \hat{S}_1 = \{\lambda_{2m+1}\}, \quad (134)$$

$$\check{S}_2 = \{\lambda_{2n+2}, \text{ with } n = 0, \dots, 2[(m-1)/2]\}, \quad \hat{S}_2 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \quad (135)$$

where (λ_{2m}) means that $\lambda_{2m} \in \hat{S}_0$ if m is odd and $\lambda_{2m} \in \hat{S}_2$ if m is even. For each p , $\hat{S}_p \cap \check{S}_p = \emptyset$, and using the product (126) one sees that the partition satisfies [compare with Eqs. (50)–(55)]

$$\begin{aligned} \check{S}_0 \cdot \hat{S}_0 &\subset \hat{S}_0, & \check{S}_1 \cdot \hat{S}_1 &\subset \hat{S}_0 \cap \hat{S}_2, \\ \check{S}_0 \cdot \hat{S}_1 &\subset \hat{S}_1, & \check{S}_1 \cdot \hat{S}_2 &\subset \hat{S}_1, \\ \check{S}_0 \cdot \hat{S}_2 &\subset \hat{S}_2, & \check{S}_2 \cdot \hat{S}_2 &\subset \hat{S}_0. \end{aligned} \quad (136)$$

Therefore

$$\check{\mathfrak{G}}_R = \check{W}_0 \oplus \check{W}_1 \oplus \check{W}_2, \quad (137)$$

corresponds to a reduced algebra of \mathfrak{G}_R , where

$$\check{W}_0 = (\check{S}_0 \times V_0) = \{\lambda_{2n}, \text{ with } n = 0, \dots, 2[m/2]\} \times \{\tilde{J}_{ab}\}, \quad (138)$$

$$\check{W}_1 = (\check{S}_1 \times V_1) = \{\lambda_{2n+1}, \text{ with } n = 0, \dots, m-1\} \times \{\tilde{Q}_\alpha\}, \quad (139)$$

$$\check{W}_2 = (\check{S}_2 \times V_2) = \{\lambda_{2n+2}, \text{ with } n = 0, \dots, 2[(m-1)/2]\} \times \{\tilde{P}_a\}. \quad (140)$$

Here, \tilde{J}_{ab} , \tilde{P}_a and \tilde{Q}_α correspond to the generators of $\mathfrak{osp}(4|1)$ superalgebra. The new superalgebra obtained by the S -expansion procedure is generated by

$$\{J_{ab}, P_a, Z_{ab}^{(k)}, \tilde{Z}_{ab}^{(k)}, Z_a^{(l)}, \tilde{Z}_a^{(l)}, Q_\alpha, \Sigma_\alpha^{(p)}\}, \quad (141)$$

where these new generators can be written as

$$\begin{aligned}
J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & \tilde{Z}_a^{(l)} &= P_{a,4l} = \lambda_{4l} \tilde{P}_a, \\
P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, & Q_\alpha &= Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, \\
Z_{ab}^{(k)} &= J_{ab,4k} = \lambda_{4k} \tilde{J}_{ab}, & \Sigma_\alpha^{(k)} &= Q_{\alpha,4k-1} = \lambda_{4k-1} \tilde{Q}_\alpha, \\
\tilde{Z}_{ab}^{(k)} &= J_{ab,4k-2} = \lambda_{4k-2} \tilde{J}_{ab}, & \Phi_\alpha^{(l)} &= Q_{\alpha,4l+1} = \lambda_{4l+1} \tilde{Q}_\alpha, \\
Z_a^{(l)} &= P_{a,4l+2} = \lambda_{4l+2} \tilde{P}_a.
\end{aligned} \tag{142}$$

with $k = 1, \dots, [\frac{m}{2}]$, $l = 1, \dots, [\frac{m-1}{2}]$. It is important to note that the super-indices k and l of spinor generators correspond to the expansion labels and they do not define an N -extended superalgebra. The N -extended case will be considered in the next section.

These new generators satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{143}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad [P_a, P_b] = Z_{ab}^{(1)}, \tag{144}$$

$$[J_{ab}, Z_{cd}^{(k)}] = \eta_{bc} Z_{ad}^{(k)} - \eta_{ac} Z_{bd}^{(k)} - \eta_{bd} Z_{ac}^{(k)} + \eta_{ad} Z_{bc}^{(k)}, \tag{145}$$

$$[Z_{ab}^{(k)}, P_c] = \eta_{bc} Z_a^{(k)} - \eta_{ac} Z_b^{(k)}, \quad [J_{ab}, Z_c^{(l)}] = \eta_{bc} Z_a^{(l)} - \eta_{ac} Z_b^{(l)}, \tag{146}$$

$$[Z_{ab}^{(k)}, Z_c^{(l)}] = \eta_{bc} Z_a^{(k+l)} - \eta_{ac} Z_b^{(k+l)}, \tag{147}$$

$$[Z_{ab}^{(k)}, Z_{cd}^{(j)}] = \eta_{bc} Z_{ad}^{(k+j)} - \eta_{ac} Z_{bd}^{(k+j)} - \eta_{bd} Z_{ac}^{(k+j)} + \eta_{ad} Z_{bc}^{(k+j)}, \tag{148}$$

$$[P_a, Z_c^{(k)}] = Z_{ab}^{(k+1)}, \quad [Z_a^{(l)}, Z_c^{(n)}] = Z_{ab}^{(l+n+1)} \tag{149}$$

$$[\tilde{Z}_{ab}^{(k)}, \tilde{Z}_{cd}^{(j)}] = \eta_{bc} \tilde{Z}_{ad}^{(k+j-1)} - \eta_{ac} \tilde{Z}_{bd}^{(k+j-1)} - \eta_{bd} \tilde{Z}_{ac}^{(k+j-1)} + \eta_{ad} \tilde{Z}_{bc}^{(k+j-1)}, \tag{150}$$

$$[J_{ab}, \tilde{Z}_{cd}^{(k)}] = \eta_{bc} \tilde{Z}_{ad}^{(k)} - \eta_{ac} \tilde{Z}_{bd}^{(k)} - \eta_{bd} \tilde{Z}_{ac}^{(k)} + \eta_{ad} \tilde{Z}_{bc}^{(k)}, \tag{151}$$

$$[\tilde{Z}_{ab}^{(k)}, P_c] = \eta_{bc} \tilde{Z}_a^{(k)} - \eta_{ac} \tilde{Z}_b^{(k)}, \quad [J_{ab}, \tilde{Z}_c^{(l)}] = \eta_{bc} \tilde{Z}_a^{(l)} - \eta_{ac} \tilde{Z}_b^{(l)}, \tag{152}$$

$$[Z_{ab}^{(k)}, \tilde{Z}_c^{(l)}] = \eta_{bc} \tilde{Z}_a^{(k+l)} - \eta_{ac} \tilde{Z}_b^{(k+l)}, \quad [\tilde{Z}_{ab}^{(k)}, Z_c^{(l)}] = \eta_{bc} \tilde{Z}_a^{(k+l)} - \eta_{ac} \tilde{Z}_b^{(k+l)}, \tag{153}$$

$$[\tilde{Z}_{ab}^{(k)}, \tilde{Z}_c^{(l)}] = \eta_{bc} \tilde{Z}_a^{(k+l-1)} - \eta_{ac} \tilde{Z}_b^{(k+l-1)}, \quad [P_a, \tilde{Z}_b^{(l)}] = \tilde{Z}_{ab}^{(l+1)} \tag{154}$$

$$[\tilde{Z}_a^{(l)}, \tilde{Z}_b^{(n)}] = Z_{ab}^{(l+n)}, \quad [Z_a^{(l)}, \tilde{Z}_b^{(n)}] = \tilde{Z}_{ab}^{(l+n+1)}, \tag{155}$$

$$[Z_{ab}^{(k)}, \tilde{Z}_{cd}^{(j)}] = \eta_{bc} \tilde{Z}_{ad}^{(k+j)} - \eta_{ac} \tilde{Z}_{bd}^{(k+j)} - \eta_{bd} \tilde{Z}_{ac}^{(k+j)} + \eta_{ad} \tilde{Z}_{bc}^{(k+j)}, \tag{156}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} Q)_\alpha, \quad [J_{ab}, \Sigma_\alpha^{(k)}] = -\frac{1}{2}(\gamma_{ab} \Sigma^{(k)})_\alpha, \tag{157}$$

$$[J_{ab}, \Phi_\alpha^{(l)}] = -\frac{1}{2}(\gamma_{ab} \Phi^{(l)})_\alpha, \quad [\tilde{Z}_{ab}^{(k)}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} \Sigma^{(k)})_\alpha, \tag{158}$$

$$[\tilde{Z}_{ab}^{(k)}, \Sigma_\alpha^{(j)}] = -\frac{1}{2}(\gamma_{ab} \Phi^{(k+j-1)})_\alpha, \quad [\tilde{Z}_{ab}^{(k)}, \Phi_\alpha^{(l)}] = -\frac{1}{2}(\gamma_{ab} \Sigma^{(k+l)})_\alpha, \tag{159}$$

$$[Z_{ab}^{(k)}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab} \Phi^{(k)})_\alpha, \quad [Z_{ab}^{(k)}, \Sigma_\alpha^{(j)}] = -\frac{1}{2}(\gamma_{ab} \Sigma^{(k+j)})_\alpha, \tag{160}$$

$$[Z_{ab}^{(k)}, \Phi_\alpha^{(l)}] = -\frac{1}{2}(\gamma_{ab} \Phi^{(k+l)})_\alpha, \quad [P_a, Q_\alpha] = -\frac{1}{2}(\gamma_a \Sigma^{(1)})_\alpha \tag{161}$$

$$[P_a, \Sigma_\alpha^{(k)}] = -\frac{1}{2}(\gamma_a \Phi^{(k)})_\alpha, \quad [P_a, \Phi_\alpha^{(l)}] = -\frac{1}{2}(\gamma_a \Sigma^{(l+1)})_\alpha, \tag{162}$$

$$[\tilde{Z}_a^{(l)}, Q_\alpha] = -\frac{1}{2}(\gamma_a \Phi^{(l)})_\alpha, \quad [\tilde{Z}_a^{(l)}, \Sigma_\alpha^{(k)}] = -\frac{1}{2}(\gamma_a \Sigma^{(l+k)})_\alpha, \tag{163}$$

$$[\tilde{Z}_a^{(l)}, \Phi_\alpha^{(n)}] = -\frac{1}{2}(\gamma_a \Phi^{(l+n)})_\alpha, \quad [Z_a^{(l)}, Q_\alpha] = -\frac{1}{2}(\gamma_a \Sigma^{(l+1)})_\alpha, \quad (164)$$

$$[Z_a^{(l)}, \Sigma_\alpha^{(n)}] = -\frac{1}{2}(\gamma_a \Phi^{(l+n)})_\alpha, \quad [Z_a^{(l)}, \Phi_\alpha^{(n)}] = -\frac{1}{2}(\gamma_a \Sigma^{(l+n+1)})_\alpha, \quad (165)$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(1)} - 2(\gamma^a C)_{\alpha\beta} P_a], \quad (166)$$

$$\{Q_\alpha, \Sigma_\beta^{(k)}\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} Z_{ab}^{(k)} - 2(\gamma^a C)_{\alpha\beta} \tilde{Z}_a^{(k)}], \quad (167)$$

$$\{Q_\alpha, \Phi_\beta^{(l)}\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(l+1)} - 2(\gamma^a C)_{\alpha\beta} Z_a^{(l)}], \quad (168)$$

$$\{\Sigma_\alpha^{(k)}, \Sigma_\beta^{(j)}\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(k+j)} - 2(\gamma^a C)_{\alpha\beta} Z_a^{(k+j-1)}], \quad (169)$$

$$\{\Sigma_\alpha^{(k)}, \Phi_\beta^{(l)}\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} Z_{ab}^{(k+l)} - 2(\gamma^a C)_{\alpha\beta} \tilde{Z}_a^{(k+l)}], \quad (170)$$

$$\{\Phi_\alpha^{(l)}, \Phi_\beta^{(n)}\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(l+n+1)} - 2(\gamma^a C)_{\alpha\beta} Z_a^{(l+n)}], \quad (171)$$

with $k, j = 1, \dots, [\frac{m}{2}]$, $l, n = 1, \dots, [\frac{m-1}{2}]$. The commutation relations can be obtained using the multiplication law of the semigroup (126) and the commutation relations of the original superalgebra (45)–(49). One sees that when $k + l > [\frac{m}{2}]$ then the generators $T_A^{(k)}$ and $T_B^{(l)}$ are abelian.

The new superalgebra obtained after a reduced resonant S -expansion of $\mathfrak{osp}(4|1)$ superalgebra corresponds to the $D = 4$ minimal Maxwell superalgebra type $s\mathcal{M}_{m+2}$. This superalgebra contains the Maxwell algebra type $\mathcal{M}_{m+2} = \{J_{ab}, P_a, Z_{ab}^{(k)}, Z_a^{(l)}\}$ as a subalgebra (Eqs. (143)–(149)) [8,9]. Interestingly, when $m = 2$ and imposing $\tilde{Z}_{ab}^{(1)} = 0$ we recover the minimal Maxwell superalgebra $s\mathcal{M}$. The case $m = 1$ corresponds to $D = 4$ Poincaré superalgebra $s\mathcal{P} = \{J_{ab}, P_a, Q_\alpha\}$. This is not a surprise since the reduced resonant $S_E^{(2)}$ -expansion of $\mathfrak{osp}(4|1)$ coincides with a Inönü–Wigner contraction.

In this case, the S -expansion method produces new Majorana spinors charge $\Sigma^{(k)}$ and $\Phi^{(l)}$. These fermionic generators transform as spinors under Lorentz transformations. One can see that the Jacobi identities for spinors generators are satisfied due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = (C\gamma^{ab})_{(\alpha\beta}(C\gamma_{ab})_{\gamma\delta)} = 0$ (cyclic permutations of α, β, γ). In fact, all the commutators satisfy the JI since they correspond to expansions of the original JI of $\mathfrak{osp}(4|1)$.

5. S -expansion of the $\mathfrak{osp}(4|N)$ superalgebra

5.1. N -extended super-Maxwell algebras

We have shown that the minimal $D = 4$ Maxwell superalgebras type $s\mathcal{M}_{m+2}$ can be obtained from a reduced resonant $S_E^{(2m)}$ -expansion of $\mathfrak{osp}(4|1)$ superalgebra. It seems natural to expect to obtain the $D = 4$ N -extended Maxwell superalgebras from an S -expansion of $\mathfrak{osp}(4|N)$ superalgebra.

If we want to apply an S -expansion, first it is convenient to decompose the original superalgebra \mathfrak{g} as a direct sum of subspaces V_p ,

$$\begin{aligned} \mathfrak{g} = \mathfrak{osp}(4|N) &= (\mathfrak{so}(3, 1) \oplus \mathfrak{so}(N)) \oplus \frac{\mathfrak{osp}(4|N)}{\mathfrak{sp}(4) \oplus \mathfrak{so}(N)} \oplus \frac{\mathfrak{sp}(4)}{\mathfrak{so}(3, 1)} \\ &= V_0 \oplus V_1 \oplus V_2, \end{aligned} \quad (172)$$

where V_0 corresponds to the subspace generated by Lorentz generators \tilde{J}_{ab} and by $\frac{N(N-1)}{2}$ internal symmetry generators T^{ij} , V_1 corresponds to the fermionic subspace generated by N Majorana spinor charges \tilde{Q}_α^i ($i = 1, \dots, N$; $\alpha = 1, \dots, 4$) and V_2 corresponds to the Ads boost generated by \tilde{P}_a . The $\mathfrak{osp}(4|N)$ (anti)commutation relations read

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \quad (173)$$

$$[T^{ij}, T^{kl}] = \delta^{jk}T^{il} - \delta^{ik}T^{jl} - \delta^{jl}T^{ik} + \delta^{il}T^{jk}, \quad (174)$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \quad (175)$$

$$[\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab}, \quad (176)$$

$$[\tilde{J}_{ab}, \tilde{Q}_\alpha^i] = -\frac{1}{2}(\gamma_{ab}\tilde{Q}^i)_\alpha, \quad [\tilde{P}_a, \tilde{Q}_\alpha^i] = -\frac{1}{2}(\gamma_a\tilde{Q}^i)_\alpha, \quad (177)$$

$$[T^{ij}, \tilde{Q}_\alpha^k] = (\delta^{jk}\tilde{Q}_\alpha^i - \delta^{ik}\tilde{Q}_\alpha^j), \quad (178)$$

$$\{\tilde{Q}_\alpha^i, \tilde{Q}_\beta^j\} = -\frac{1}{2}\delta^{ij}[(\gamma^{ab}C)_{\alpha\beta}\tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta}\tilde{P}_a] + C_{\alpha\beta}T^{ij}, \quad (179)$$

where $i, j, k, l = 1, \dots, N$.

The subspace structure may be written as

$$[V_0, V_0] \subset V_0, \quad (180)$$

$$[V_0, V_1] \subset V_1, \quad (181)$$

$$[V_0, V_2] \subset V_2, \quad (182)$$

$$[V_1, V_1] \subset V_0 \oplus V_2, \quad (183)$$

$$[V_1, V_2] \subset V_1, \quad (184)$$

$$[V_2, V_2] \subset V_0. \quad (185)$$

Let us consider $S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ as the relevant finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{cuando } \alpha + \beta \leq 5, \\ \lambda_5, & \text{cuando } \alpha + \beta > 5. \end{cases} \quad (186)$$

In this case, λ_5 plays the role of the zero element of the semigroup $S_E^{(4)}$.

Let $S_E^{(4)} = S_0 \cup S_1 \cup S_2$ be a subset decomposition of $S_E^{(4)}$ with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}, \quad (187)$$

$$S_1 = \{\lambda_1, \lambda_3, \lambda_5\}, \quad (188)$$

$$S_2 = \{\lambda_2, \lambda_4, \lambda_5\}, \quad (189)$$

This subset decomposition satisfies the resonance condition since we have [compare with Eqs. (180)–(185)]

$$\begin{aligned} S_0 \cdot S_0 &\subset S_0, & S_1 \cdot S_1 &\subset S_0 \cap S_2, \\ S_0 \cdot S_1 &\subset S_1, & S_1 \cdot S_2 &\subset S_1, \\ S_0 \cdot S_2 &\subset S_2, & S_2 \cdot S_2 &\subset S_0. \end{aligned} \quad (190)$$

Thus, according to Theorem IV.2 of Ref. [5], we have that

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \quad (191)$$

is a resonant subalgebra of $S_E^{(4)} \times \mathfrak{g}$, where

$$\begin{aligned} W_0 &= (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \times \{\tilde{J}_{ab}, T^{ij}\} \\ &= \{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_4 \tilde{J}_{ab}, \lambda_5 \tilde{J}_{ab}, \lambda_0 T^{ij}, \lambda_2 T^{ij}, \lambda_4 T^{ij}, \lambda_5 T^{ij}\}, \end{aligned} \quad (192)$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3, \lambda_5\} \times \{\tilde{Q}_\alpha\} = \{\lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\alpha, \lambda_5 \tilde{Q}_\alpha\}, \quad (193)$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2, \lambda_4, \lambda_5\} \times \{\tilde{P}_a\} = \{\lambda_2 \tilde{P}_a, \lambda_4 \tilde{P}_a, \lambda_5 \tilde{P}_a\}. \quad (194)$$

Imposing $\lambda_5 T_A = 0$, the 0_S -reduced resonant superalgebra is obtained. The new superalgebra is generated by $\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a, Q_\alpha^i, \Sigma_\alpha^i, T^{ij}, Y^{ij}, \tilde{Y}^{ij}\}$ where the new generators can be written as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & Q_\alpha^i &= Q_{\alpha,1}^i = \lambda_1 \tilde{Q}_\alpha^i, \\ P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, & \Sigma_\alpha^i &= \Sigma_{\alpha,3}^i = \lambda_3 \tilde{Q}_\alpha^i, \\ Z_{ab} &= J_{ab,4} = \lambda_4 \tilde{J}_{ab}, & T^{ij} &= T^{ij}_{,0} = \lambda_0 T^{ij}, \\ \tilde{Z}_{ab} &= J_{ab,2} = \lambda_2 \tilde{J}_{ab}, & Y^{ij} &= T^{ij}_{,4} = \lambda_4 T^{ij}, \\ \tilde{Z}_a &= P_{a,4} = \lambda_4 \tilde{P}_a, & \tilde{Y}^{ij} &= T^{ij}_{,2} = \lambda_2 T^{ij}. \end{aligned} \quad (195)$$

Then using the multiplication law of the semigroup (186) and the commutations relations of the original superalgebra (173)–(179) it is possible to write the resulting superalgebra as

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (196)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad [P_a, P_b] = Z_{ab}, \quad (197)$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (198)$$

$$[J_{ab}, \tilde{Z}_{cd}] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \quad (199)$$

$$[\tilde{Z}_{ab}, \tilde{Z}_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (200)$$

$$[J_{ab}, \tilde{Z}_c] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \quad (201)$$

$$[\tilde{Z}_{ab}, P_c] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \quad (202)$$

$$[T^{ij}, T^{kl}] = \delta^{jk} T^{il} - \delta^{ik} T^{jl} - \delta^{jl} T^{ik} + \delta^{il} T^{jk}, \quad (203)$$

$$[T^{ij}, Y^{kl}] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \quad (204)$$

$$[T^{ij}, \tilde{Y}^{kl}] = \delta^{jk} \tilde{Y}^{il} - \delta^{ik} \tilde{Y}^{jl} - \delta^{jl} \tilde{Y}^{ik} + \delta^{il} \tilde{Y}^{jk}, \quad (205)$$

$$[\tilde{Y}^{ij}, \tilde{Y}^{kl}] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \quad (206)$$

$$[J_{ab}, Q_\alpha^i] = -\frac{1}{2}(\gamma_{ab} Q_\alpha^i)_\alpha, \quad [\tilde{Z}_{ab}, Q_\alpha^i] = -\frac{1}{2}(\gamma_{ab} \Sigma_\alpha^i)_\alpha, \quad (207)$$

$$[J_{ab}, \Sigma_\alpha^i] = -\frac{1}{2}(\gamma_{ab} \Sigma_\alpha^i)_\alpha, \quad [T^{ij}, Q_\alpha^i] = (\delta^{jk} Q_\alpha^i - \delta^{ik} Q_\alpha^j), \quad (208)$$

$$[T^{ij}, \Sigma_\alpha^k] = (\delta^{jk} \Sigma_\alpha^i - \delta^{ik} \Sigma_\alpha^j), \quad (209)$$

$$[\tilde{Y}^{ij}, Q_\alpha^k] = (\delta^{jk} \Sigma_\alpha^i - \delta^{ik} \Sigma_\alpha^j), \quad (210)$$

$$[P_a, Q_\alpha^i] = -\frac{1}{2}(\gamma_a \Sigma_\alpha^i)_\alpha, \quad (211)$$

$$\{Q_\alpha^i, Q_\beta^j\} = -\frac{1}{2}\delta^{ij}[(\gamma^{ab}C)_{\alpha\beta}\tilde{Z}_{ab} - 2(\gamma^a C)_{\alpha\beta}P_a] + C_{\alpha\beta}\tilde{Y}^{ij}, \quad (212)$$

$$\{Q_\alpha^i, \Sigma_\beta^j\} = -\frac{1}{2}\delta^{ij}[(\gamma^{ab}C)_{\alpha\beta}Z_{ab} - 2(\gamma^a C)_{\alpha\beta}\tilde{Z}_a] + C_{\alpha\beta}Y^{ij}, \quad (213)$$

$$\text{others} = 0. \quad (214)$$

The new superalgebra obtained after a reduced resonant $S_E^{(4)}$ -expansion of $\mathfrak{osp}(4|N)$ superalgebra corresponds to the $D = 4$ N -extended Maxwell superalgebra $s\mathcal{M}_4^{(N)}$. An alternative expansion procedure to obtain the N -extended Maxwell superalgebra has been proposed in Ref. [19]. Interestingly, this superalgebra contains the generalized Maxwell algebra $g\mathcal{M} = \{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a\}$ as a subalgebra (see Appendix B). One sees that the S -expansion procedure introduces additional bosonic generators which modify the minimal Maxwell superalgebra [see Eqs. (212), (213)]. Naturally when $\tilde{Z}_a = \tilde{Z}_{ab} = Y^{ij} = \tilde{Y}^{ij} = 0$, we obtain the simplest $D = 4$ N -extended Maxwell superalgebra $s\mathcal{M}^{(N)}$ generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha^i, \Sigma_\alpha^i, T_{ab}\}$. Eventually for $N = 1$, with $T_{ab} = 0$, the $D = 4$ minimal Maxwell superalgebra $s\mathcal{M}$ is recovered. It is important to note that setting some generators equals to zero does not always lead to a Lie superalgebra. Nevertheless, the properties of the gamma matrices in 4 dimensions permit us to impose some generators equals to zero without breaking the Jacobi Identity.

We can generalize this procedure and obtain the N -extended super-Maxwell algebra type $s\mathcal{M}_{m+2}^{(N)}$ as an reduced resonant S -expansion of $\mathfrak{osp}(4|N)$ with $S_E^{(2m)} = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{2m+1}\}$ as abelian semigroup. In fact, if we consider a resonant subset decomposition $S_E^{(2m)} = S_0 \cup S_1 \cup S_2$, where

$$S_p = \left\{ \lambda_{2n+p}, \text{ with } n = 0, \dots, \left\lfloor \frac{2m-p}{2} \right\rfloor \right\} \cup \{\lambda_{2m+1}\}, \quad p = 0, 1, 2, \quad (215)$$

and let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_{2n}, \text{ with } n = 0, \dots, 2[m/2]\}, \quad \hat{S}_0 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \quad (216)$$

$$\check{S}_1 = \{\lambda_{2n+1}, \text{ with } n = 0, \dots, m-1\}, \quad \hat{S}_1 = \{\lambda_{2m+1}\}, \quad (217)$$

$$\check{S}_2 = \{\lambda_{2n+2}, \text{ with } n = 0, \dots, 2[(m-1)/2]\}, \quad \hat{S}_2 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \quad (218)$$

where (λ_{2m}) means that $\lambda_{2m} \in \hat{S}_0$ if m is odd and $\lambda_{2m} \in \hat{S}_2$ if m is even. This decomposition satisfies the resonant condition for any value of m and we find that

$$\check{\mathfrak{O}}_R = (\check{S}_0 \times V_0) \oplus (\check{S}_1 \times V_1) \oplus (\check{S}_2 \times V_2), \quad (219)$$

corresponds to a reduced resonant algebra. This new superalgebra correspond to the N -extended Maxwell superalgebra type $s\mathcal{M}_{m+2}^{(N)}$ which is generated by

$$\{J_{ab}, P_a, Z_{ab}^{(k)}, \tilde{Z}_{ab}^{(k)}, Z_a^{(k)}, \tilde{Z}_a^{(k)}, Q_\alpha^i, \Sigma_\alpha^{i(k)}, \Phi_\alpha^{i(k)}, T^{ij}, Y^{ij(k)}, \tilde{Y}^{ij(k)}\}. \quad (220)$$

These generators can be written as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, \\ Z_{ab}^{(k)} &= J_{ab,4k} = \lambda_{4k} \tilde{J}_{ab}, & \tilde{Z}_{ab}^{(k)} &= J_{ab,4k-2} = \lambda_{4k-2} \tilde{J}_{ab}, \\ Z_a^{(l)} &= P_{a,4l+2} = \lambda_{4l+2} \tilde{P}_a, & \tilde{Z}_a^{(l)} &= P_{a,4l} = \lambda_{4l} \tilde{P}_a, \\ Q_\alpha^i &= Q_{\alpha,1}^i = \lambda_1 \tilde{Q}_\alpha^i, & \Sigma_\alpha^{i(k)} &= Q_{\alpha,4k-1}^i = \lambda_{4k-1} \tilde{Q}_\alpha^i, \end{aligned}$$

$$\begin{aligned}\Phi_{\alpha}^{i(l)} &= Q_{\alpha,4l+1}^i = \lambda_{4l+1} \tilde{Q}_{\alpha}^i, & T^{ij} &= T^{ij}_{,0} = \lambda_0 T^{ij}, \\ Y^{ij(k)} &= T^{ij}_{,4k} = \lambda_{4k} T^{ij}, & \tilde{Y}^{ij(k)} &= T^{ij}_{,4k-2} = \lambda_{4k-2} T^{ij},\end{aligned}\quad (221)$$

with $k = 1, \dots, [\frac{m}{2}]$, $l = 1, \dots, [\frac{m-1}{2}]$, $i, j = 1, \dots, N$. The new bosonics generators $\{Z_{ab}, \tilde{Z}_{ab}, Z_a, \tilde{Z}_a, Y^{ij}, \tilde{Y}^{ij}\}$ modify some anticommutators of the minimal Maxwell superalgebra type (166)–(171). Now we have

$$\{Q_{\alpha}^i, Q_{\beta}^j\} = -\frac{1}{2} \delta^{ij} [(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(1)} - 2(\gamma^a C)_{\alpha\beta} P_a] + C_{\alpha\beta} \tilde{Y}^{ij(1)}, \quad (222)$$

$$\{Q_{\alpha}^i, \Sigma_{\beta}^{j(k)}\} = -\frac{1}{2} \delta^{ij} [(\gamma^{ab} C)_{\alpha\beta} Z_{ab}^{(k)} - 2(\gamma^a C)_{\alpha\beta} \tilde{Z}_a^{(k)}] + C_{\alpha\beta} Y^{ij(k)}, \quad (223)$$

$$\{Q_{\alpha}^i, \Phi_{\beta}^{j(l)}\} = -\frac{1}{2} \delta^{ij} [(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(l+1)} - 2(\gamma^a C)_{\alpha\beta} Z_a^{(l)}] + C_{\alpha\beta} \tilde{Y}^{ij(l+1)}, \quad (224)$$

$$\{\Sigma_{\alpha}^{i(k)}, \Sigma_{\beta}^{j(q)}\} = -\frac{1}{2} \delta^{ij} [(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(k+q)} - 2(\gamma^a C)_{\alpha\beta} Z_a^{(k+q-1)}] + C_{\alpha\beta} \tilde{Y}^{ij(k+q)}, \quad (225)$$

$$\{\Sigma_{\alpha}^{i(k)}, \Phi_{\beta}^{j(l)}\} = -\frac{1}{2} \delta^{ij} [(\gamma^{ab} C)_{\alpha\beta} Z_{ab}^{(k+l)} - 2(\gamma^a C)_{\alpha\beta} \tilde{Z}_a^{(k+l)}] + C_{\alpha\beta} Y^{ij(k+l)}, \quad (226)$$

$$\{\Phi_{\alpha}^{i(l)}, \Phi_{\beta}^{j(n)}\} = -\frac{1}{2} \delta^{ij} [(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab}^{(l+n+1)} - 2(\gamma^a C)_{\alpha\beta} Z_a^{(l+n)}] + C_{\alpha\beta} \tilde{Y}^{ij(l+n+1)}, \quad (227)$$

with $k, q = 1, \dots, [\frac{m}{2}]$, $l, n = 1, \dots, [\frac{m-1}{2}]$, $i, j = 1, \dots, N$. The internal symmetries generators also brings some new commutation relations besides the commutators (143)–(165),

$$[T^{ij}, T^{gh}] = \delta^{jg} T^{ih} - \delta^{ig} T^{jh} - \delta^{jh} T^{ig} + \delta^{ih} T^{jg}, \quad (228)$$

$$[T^{ij}, Y^{gh(k)}] = \delta^{jg} Y^{ih(k)} - \delta^{ig} Y^{jh(k)} - \delta^{jh} Y^{ig(k)} + \delta^{ih} Y^{jg(k)}, \quad (229)$$

$$[T^{ij}, \tilde{Y}^{gh(k)}] = \delta^{jg} \tilde{Y}^{ih(k)} - \delta^{ig} \tilde{Y}^{jh(k)} - \delta^{jh} \tilde{Y}^{ig(k)} + \delta^{ih} \tilde{Y}^{jg(k)}, \quad (230)$$

$$\begin{aligned}[\tilde{Y}^{ij(k)}, \tilde{Y}^{gh(q)}] &= \delta^{jg} Y^{ih(k+q-1)} - \delta^{ig} Y^{jh(k+q-1)} - \delta^{jh} Y^{ig(k+q-1)} \\ &\quad + \delta^{ih} Y^{jg(k+q-1)},\end{aligned}\quad (231)$$

$$[\tilde{Y}^{ij(k)}, Y^{gh(q)}] = \delta^{jg} \tilde{Y}^{ih(k+q)} - \delta^{ig} \tilde{Y}^{jh(k+q)} - \delta^{jh} \tilde{Y}^{ig(k+q)} + \delta^{ih} \tilde{Y}^{jg(k+q)}, \quad (232)$$

$$[Y^{ij(k)}, Y^{gh(q)}] = \delta^{jg} Y^{ih(k+q)} - \delta^{ig} Y^{jh(k+q)} - \delta^{jh} Y^{ig(k+q)} + \delta^{ih} Y^{jg(k+q)}, \quad (233)$$

$$[T^{ij}, Q_{\alpha}^i] = (\delta^{jk} Q_{\alpha}^i - \delta^{ik} Q_{\alpha}^j), \quad (234)$$

$$[T^{ij}, \Sigma_{\alpha}^{g(k)}] = [\tilde{Y}^{ij(k)}, Q_{\alpha}^g] = (\delta^{jg} \Sigma_{\alpha}^{i(k)} - \delta^{ig} \Sigma_{\alpha}^{i(k)}), \quad (235)$$

$$[T^{ij}, \Phi_{\alpha}^{g(k)}] = [Y^{ij(k)}, Q_{\alpha}^g] = (\delta^{jg} \Phi_{\alpha}^{i(k)} - \delta^{ig} \Phi_{\alpha}^{i(k)}), \quad (236)$$

$$[\tilde{Y}^{ij(k)}, \Phi_{\alpha}^{g(q)}] = [Y^{ij(k)}, \Sigma_{\alpha}^{g(q)}] = (\delta^{jg} \Sigma_{\alpha}^{i(k+q)} - \delta^{ig} \Sigma_{\alpha}^{i(k+q)}), \quad (237)$$

$$[\tilde{Y}^{ij(k)}, \Sigma_{\alpha}^{g(q)}] = (\delta^{jg} \Phi_{\alpha}^{i(k+q-1)} - \delta^{ig} \Phi_{\alpha}^{i(k+q-1)}), \quad (238)$$

$$[Y^{ij(k)}, \Phi_{\alpha}^{g(q)}] = (\delta^{jg} \Phi_{\alpha}^{i(k+q)} - \delta^{ig} \Phi_{\alpha}^{i(k+q)}). \quad (239)$$

The commutation relations can be obtained using the multiplication law of the semigroup and the commutation relations of the $\mathfrak{osp}(4|N)$ superalgebra. As in the case of minimal super-Maxwell algebra type one sees that when $k + q > [\frac{m}{2}]$ then the generators $T_A^{(k)}$ and $T_B^{(q)}$ are abelian. As in the previous case, the S -expansion method produces new Majorana spinors charge $\Sigma^{i(k)}$ and $\Phi^{i(l)}$ which transform as spinors under Lorentz transformations.

The N -extended Maxwell superalgebra type $s\mathcal{M}_{m+2}^{(N)}$ contains the Maxwell algebra type $\mathcal{M}_{m+2} = \{J_{ab}, P_a, Z_{ab}^{(k)}, Z_a^{(l)}\}$ as a subalgebra (Eqs. (143)–(149)) [9]. We can see that for $m = 2$ we recover the $D = 4$ N -extended Maxwell superalgebra $s\mathcal{M}_4^{(N)}$. It is interesting to observe that for $m = 1$ we obtain the $D = 4$ N -extended Poincaré superalgebra $s\mathcal{P}^{(N)} = \{J_{ab}, P_a, Q_\alpha, T^{ij}\}$. This is not a surprise because the reduced resonant $S_E^{(2)}$ -expansion of $\mathfrak{osp}(4|N)$ coincides with an Inönü–Wigner contraction.

Interestingly, it is possible to write the N -extended Maxwell superalgebra type $s\mathcal{M}_{m+2}^{(N)}$ in a very compact way defining

$$\begin{aligned} J_{ab,(k)} &= \lambda_{2k} \tilde{J}_{ab}, \\ P_{a,(l)} &= \lambda_{2l} \tilde{P}_a, \\ Q_{\alpha,(p)} &= \lambda_{2p-1} \tilde{Q}_\alpha, \\ Y_{(k)}^{ij} &= \lambda_{2k} T^{ij} \end{aligned}$$

with $k = 0, \dots, m-1$; $l = 1, \dots, m$; $p = 1, \dots, m$ when m is odd and $k = 0, \dots, m$; $l = 1, \dots, m-1$; $p = 1, \dots, m$ when m is even. Here, the generators \tilde{J}_{ab} , \tilde{P}_a , \tilde{Q}_α and T^{ij} correspond to the $\mathfrak{osp}(4|N)$ generators. Then using the multiplication law of the semigroup (126) and the commutations relations of the original superalgebra (173)–(179) is possible to write the resulting superalgebra as

$$[J_{ab,(k)}, J_{cd,(j)}] = \eta_{bc} J_{ad,(k+j)} - \eta_{ac} J_{bd,(k+j)} - \eta_{bd} J_{ac,(k+j)} + \eta_{ad} J_{bc,(k+j)}, \quad (240)$$

$$[Y_{(k)}^{ij}, Y_{(j)}^{gh}] = \delta^{jg} Y_{(k+j)}^{ih} - \delta^{ig} Y_{(k+j)}^{jh} - \delta^{jh} Y_{(k+j)}^{ig} + \delta^{ih} Y_{(k+j)}^{jg}, \quad (241)$$

$$[J_{ab,(k)}, P_{c,(l)}] = \eta_{bc} P_{a,(k+l)} - \eta_{ac} P_{b,(k+l)}, \quad (242)$$

$$[P_{a,(l)}, P_{b,(n)}] = J_{ab,(l+n)}, \quad (243)$$

$$[J_{ab,(k)}, Q_{\alpha,(p)}] = -\frac{1}{2}(\gamma_{ab} Q)_{\alpha,(k+p)}, \quad (244)$$

$$[P_{a,(l)}, Q_{\alpha,(p)}] = -\frac{1}{2}(\gamma_a Q)_{\alpha,(l+p)}, \quad (245)$$

$$[T_{(k)}^{ij}, Q_{\alpha,(p)}^g] = (\delta^{jg} Q_{\alpha,(k+p)}^i - \delta^{ig} Q_{\alpha,(k+p)}^j), \quad (246)$$

$$\{Q_{\alpha,(p)}, Q_{\beta,(q)}\} = -\frac{1}{2}[(\gamma^{ab} C)_{\alpha\beta} J_{ab,(p+q)} - 2(\gamma^a C)_{\alpha\beta} P_{a,(p+q)}] + C_{\alpha\beta} Y_{(p+q)}^{ij}, \quad (247)$$

where $i, j, g, h = 1, \dots, N$. Naturally, when $k + j > m$ then the generators $T_A^{(k)}$ and $T_B^{(j)}$ are abelian. With this notation it is not trivial to see the Maxwell algebra type \mathcal{M}_{m+2} as a subalgebra. However it could be useful in order to construct an action for this superalgebra.

6. Comments and possible developments

In the present work we have shown that the Maxwell superalgebras found by the MC expansion method in Ref. [19] can be derived alternatively by the S -expansion procedure. In particular, the S -expansion of $\mathfrak{osp}(4|1)$ permits us to obtain the minimal Maxwell superalgebra $s\mathcal{M}$. Then choosing different semigroups we have shown that it is possible to define new minimal $D = 4$ Maxwell superalgebras type $s\mathcal{M}_{m+2}$ which can be seen as a generalization of the D'Auria–Fré superalgebra and the Green algebras introduced in Refs. [7,22], respectively. Interestingly, the

case $m = 1$ corresponds to the minimal Poincaré superalgebra. Recently it was shown that the minimal Maxwell superalgebra $s\mathcal{M}$ may be used to obtain the minimal $D = 4$ pure supergravity [21]. It seems that the new minimal Maxwell superalgebras $s\mathcal{M}_{m+2}$ defined here may be good candidates to enlarge the $D = 4$ pure supergravity Lagrangian leading to a generalized cosmological term. Interestingly we have shown that this Maxwell superalgebra contains the Maxwell algebras type \mathcal{M}_{m+2} as bosonic subalgebras.

We also have shown that the $D = 4$ N -extended Maxwell superalgebra $s\mathcal{M}^{(N)}$, derived initially as a MC expansion in Ref. [19], can be obtained alternatively as an S -expansion of $\mathfrak{osp}(4|N)$. In this case the S -expansion produces additional bosonic generators which modify the minimal Maxwell superalgebra. Choosing bigger semigroups we have shown that it is possible to define new $D = 4$ N -extended Maxwell superalgebras type $s\mathcal{M}_{m+2}^{(N)}$. Naturally when $m = 2$ we recover the $s\mathcal{M}^{(N)}$ superalgebra and for $N = 1$ we recover the Maxwell algebra type $s\mathcal{M}_{m+2}$. It would be interesting to build Lagrangians with the 2-form curvature associated to these new N -extended Maxwell superalgebras $s\mathcal{M}_{m+2}^{(N)}$ and study their relation with the N -extended supergravity in $D = 4$ [work in progress].

Thus, we have shown that the S -expansion procedure is a powerful and simple tool in order to derive new Lie superalgebras. In fact, the introduction of new Majorana spinor charges could not be guessed trivially. The method considered here could play an important role in the context of supergravity in higher dimensions. It seems that it should be possible to recover standard odd- and even-dimensional supergravity from the Maxwell superalgebra family [work in progress].

Acknowledgements

The authors wish to thank L. Andrianopoli, R. D'Auria and M. Trigiante for enlightening discussions and their kind hospitality at Dipartimento Scienza Applicata e Tecnologia of Politecnico di Torino, where this work was done. We are also grateful to P. Salgado for introducing the topics covered in the present work. The authors also thank O. Fierro for helpful comments on the subject of this work. The authors were supported by grants from the Comisión Nacional de Investigación Científica y Tecnológica (CONICYT) and from the Universidad de Concepción, Chile. This work was supported in part by FONDECYT Grants No. 1130653.

Appendix A. S -expansion of the commutation relations

Let \mathfrak{g} be a Lie (super)algebra given by

$$[T_A, T_B] = C_{AB}^C T_C.$$

Let $S = \{\lambda_\alpha\}$ be an abelian semigroup with 2-selector $K_{\alpha\beta}^\gamma$. Let us denote a basis element of the direct product $S \times \mathfrak{g}$ by $T_{(A,\alpha)} = \lambda_\alpha T_A$ and consider the induced commutator $[T_{(A,\alpha)}, T_{(B,\beta)}] = \lambda_\alpha \lambda_\beta [T_A, T_B]$. Then $S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants

$$C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}^\gamma C_{AB}^C. \quad (248)$$

Let us consider as example the $S_E^{(4)}$ -expansion of the anticommutator of $\mathfrak{osp}(4|1)$,

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = -\frac{1}{2}[(\gamma^{ab}C)_{\alpha\beta} \tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta} \tilde{P}_a]. \quad (249)$$

We have said that a decomposition of the original algebra \mathfrak{g} is given by

$$\begin{aligned} \mathfrak{g} = \mathfrak{osp}(4|1) &= \mathfrak{so}(3, 1) \oplus \frac{\mathfrak{osp}(4|1)}{\mathfrak{sp}(4)} \oplus \frac{\mathfrak{sp}(4)}{\mathfrak{so}(3, 1)} \\ &= V_0 \oplus V_1 \oplus V_2, \end{aligned} \quad (250)$$

Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ where

$$\check{S}_0 = \{\lambda_0, \lambda_4, \lambda_2\}, \quad \hat{S}_0 = \{\lambda_5\}, \quad (251)$$

$$\check{S}_1 = \{\lambda_1, \lambda_3\}, \quad \hat{S}_1 = \{\lambda_5\}, \quad (252)$$

$$\check{S}_2 = \{\lambda_2\}, \quad \hat{S}_2 = \{\lambda_4, \lambda_5\}. \quad (253)$$

Then, we have said that

$$\check{\mathfrak{O}}_R = (\check{S}_0 \times V_0) \oplus (\check{S}_1 \times V_1) \oplus (\check{S}_2 \times V_2), \quad (254)$$

corresponds to a reduced resonant superalgebra. Thus the new Majorana spinor charges are given by

$$Q_\alpha = Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, \quad (255)$$

$$\Sigma_\alpha = Q_{\alpha,3} = \lambda_3 \tilde{Q}_\alpha, \quad (256)$$

where \tilde{Q}_α corresponds to the original Majorana spinor charge. Then, the new anticommutators are given by

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{\lambda_1 \tilde{Q}_\alpha, \lambda_1 \tilde{Q}_\beta\} \\ &= \lambda_1 \lambda_1 \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} \\ &= -\lambda_2 \frac{1}{2} [(\gamma^{ab} C)_{\alpha\beta} \tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta} \tilde{P}_a] \\ &= -\frac{1}{2} [(\gamma^{ab} C)_{\alpha\beta} \lambda_2 \tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta} \lambda_2 \tilde{P}_a] \\ &= -\frac{1}{2} [(\gamma^{ab} C)_{\alpha\beta} \tilde{Z}_{ab} - 2(\gamma^a C)_{\alpha\beta} P_a], \end{aligned} \quad (257)$$

where we have used that $\tilde{Z}_{ab} = J_{ab,2} = \lambda_2 \tilde{J}_{ab}$ and $P_a = P_{a,2} = \lambda_2 \tilde{P}_a$. In the same way, it is possible to show that

$$\begin{aligned} \{Q_\alpha, \Sigma_\beta\} &= \{\lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\beta\} \\ &= -\lambda_4 \frac{1}{2} [(\gamma^{ab} C)_{\alpha\beta} \tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta} \tilde{P}_a] \\ &= -\frac{1}{2} [(\gamma^{ab} C)_{\alpha\beta} \lambda_4 \tilde{J}_{ab} - 2(\gamma^a C)_{\alpha\beta} \lambda_4 \tilde{P}_a] \\ &= -\frac{1}{2} [(\gamma^{ab} C)_{\alpha\beta} Z_{ab}], \end{aligned} \quad (258)$$

where we have used that $Z_{ab} = J_{ab,4} = \lambda_4 \tilde{J}_{ab}$. This procedure can be extended to any (anti)commutator of a S -expanded (super)algebra.

Appendix B. Generalized Maxwell algebra in $D = 4$ as an S -expansion

In this appendix we will show how to obtain the generalized Maxwell algebra $g\mathcal{M}$ from $\mathfrak{so}(3, 2)$ using the S -expansion procedure.

As in the previous cases, it is necessary to consider a subspaces decomposition of the original algebra $\mathfrak{so}(3, 2)$,

$$\mathfrak{g} = \mathfrak{so}(3, 2) = \mathfrak{so}(3, 1) \oplus \frac{\mathfrak{so}(3, 2)}{\mathfrak{so}(3, 1)} = V_0 \oplus V_1, \quad (259)$$

where V_0 is generated by the Lorentz generator \tilde{J}_{ab} and V_1 is generated by the Ads boost generator \tilde{P}_a . The \tilde{J}_{ab} , \tilde{P}_a generators satisfy the commutations relations (16)–(18), thus the subspace structure may be written as

$$[V_0, V_0] \subset V_0, \quad (260)$$

$$[V_0, V_1] \subset V_1, \quad (261)$$

$$[V_1, V_1] \subset V_0. \quad (262)$$

Let $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ be a finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 3, \\ \lambda_3, & \text{when } \alpha + \beta > 3. \end{cases} \quad (263)$$

Here λ_3 plays the role of the zero element of the semigroup $S_E^{(2)}$. Let us consider a subset decomposition $S_E^{(2)} = S_0 \cup S_1$, with

$$S_0 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}, \quad (264)$$

$$S_1 = \{\lambda_1, \lambda_2, \lambda_3\}, \quad (265)$$

This subset decomposition is said to be “resonant” because it satisfies [compare with Eqs. (260)–(262).]

$$S_0 \cdot S_0 \subset S_0, \quad (266)$$

$$S_0 \cdot S_1 \subset S_1, \quad (267)$$

$$S_1 \cdot S_1 \subset S_0. \quad (268)$$

Imposing the 0_S -reduction condition,

$$\lambda_3 T_A = 0, \quad (269)$$

we find a new Lie algebra generated by $\{J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a\}$ where we have defined

$$J_{ab} = J_{ab,0} = \lambda_0 \tilde{J}_{ab}, \quad (270)$$

$$P_a = P_{a,1} = \lambda_1 \tilde{P}_a, \quad (271)$$

$$Z_{ab} = J_{ab,2} = \lambda_2 \tilde{J}_{ab}, \quad (272)$$

$$\tilde{Z}_{ab} = J_{ab,1} = \lambda_1 \tilde{J}_{ab}, \quad (273)$$

$$\tilde{Z}_a = P_{a,2} = \lambda_2 \tilde{P}_a. \quad (274)$$

These new generators satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \quad (275)$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad (276)$$

$$[P_a, P_b] = Z_{ab}, \quad (277)$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (278)$$

$$[J_{ab}, \tilde{Z}_{cd}] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \quad (279)$$

$$[\tilde{Z}_{ab}, \tilde{Z}_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \quad (280)$$

$$[J_{ab}, \tilde{Z}_c] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \quad (281)$$

$$[\tilde{Z}_{ab}, P_c] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \quad (282)$$

$$\text{others} = 0, \quad (283)$$

where we have used the multiplication law of the semigroup (263) and the commutation relations of the original algebra. The new algebra obtained after a O_S -reduced resonant S -expansion of $\mathfrak{so}(3, 2)$ corresponds to a generalized Maxwell algebra $g\mathcal{M}$ in $D = 4$ [19]. This new algebra contains the Maxwell algebra \mathcal{M} as a subalgebra. It is interesting to observe that the $g\mathcal{M}$ algebra is very similar to the Maxwell algebra type \mathcal{M}_6 introduced in Refs. [8,9]. In fact, one could identify Z_{ab} , \tilde{Z}_{ab} and \tilde{Z}_a with $Z_{ab}^{(1)}$, $Z_{ab}^{(2)}$ and Z_a of \mathcal{M}_6 respectively. However, the commutation relations (277), (280) and (282) are subtly different of those of Maxwell algebra type \mathcal{M}_6 .

References

- [1] M. Hatsuda, M. Sakaguchi, Wess–Zumino term for the AdS superstring and generalized Inonu–Wigner contraction, *Prog. Theor. Phys.* 109 (2003) 853, arXiv:hep-th/0106114.
- [2] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Generating Lie and gauge free differential (super)algebras by expanding Maurer–Cartan forms and Chern–Simons supergravity, *Nucl. Phys. B* 662 (2003) 185, arXiv:hep-th/0212347.
- [3] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Extensions, expansions, Lie algebra cohomology and enlarged superspaces, *Class. Quantum Gravity* 21 (2004) S1375, arXiv:hep-th/0401033.
- [4] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Expansions of algebras and superalgebras and some applications, *Int. J. Theor. Phys.* 46 (2007) 2738, arXiv:hep-th/0703017.
- [5] F. Izaurieta, E. Rodríguez, P. Salgado, Expanding Lie (super)algebras through Abelian semigroups, *J. Math. Phys.* 47 (2006) 123512, arXiv:hep-th/0606215.
- [6] F. Izaurieta, E. Rodríguez, P. Salgado, Eleven-dimensional gauge theory for the M algebra as an Abelian semigroup expansion of $\mathfrak{osp}(32|1)$, *Eur. Phys. J. C* 54 (2008) 675, arXiv:hep-th/0606225.
- [7] R. D’Auria, P. Fré, Geometric supergravity in $d = 11$ and its hidden supergroup, *Nucl. Phys. B* 201 (1982) 101.
- [8] F. Izaurieta, P. Minning, A. Perez, E. Rodríguez, P. Salgado, Standard General relativity from Chern–Simons gravity, *Phys. Lett. B* 678 (2009) 213, arXiv:0905.2187 [hep-th].
- [9] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez, P. Salgado, Chern–Simons and Born–Infeld gravity theories and Maxwell algebras type, *Eur. Phys. J. C* 74 (2014) 2741, arXiv:1402.0023 [hep-th].
- [10] P. Salgado, S. Salgado, $\mathfrak{so}(D-1, 1) \otimes \mathfrak{so}(D-1, 2)$ algebras and gravity, *Phys. Lett. B* 728 (2014) 5.
- [11] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez, P. Salgado, Even-dimensional general relativity from Born–Infeld gravity, *Phys. Lett. B* 725 (2013) 419, arXiv:1309.0062 [hep-th].
- [12] R. Schrader, The maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields, *Fortschr. Phys.* 20 (1972) 701.
- [13] D.V. Soroka, V.A. Soroka, Tensor extension of the Poincaré algebra, *Phys. Lett. B* 607 (2005) 302, arXiv:hep-th/0410012.
- [14] J. Gomis, K. Kamimura, J. Lukierski, Deformations of Maxwell algebra and their dynamical realizations, *J. High Energy Phys.* 08 (2009) 039, arXiv:0906.4464 [hep-th].
- [15] S. Bonanos, J. Gomis, K. Kamimura, J. Lukierski, Maxwell superalgebra and superparticle in constant gauge backgrounds, *Phys. Rev. Lett.* 104 (2010) 090401, arXiv:0911.5072 [hep-th].

- [16] G.W. Gibbons, J. Gomis, C.N. Pope, Deforming the Maxwell–Sim algebra, *Phys. Rev. D* 82 (2010) 065002, arXiv:0910.3220 [hep-th].
- [17] J.A. de Azcarraga, K. Kamimura, J. Lukierski, Generalized cosmological term from Maxwell symmetries, *Phys. Rev. D* 83 (2011) 124036, arXiv:1012.4402 [hep-th].
- [18] K. Kamimura, J. Lukierski, Supersymmetrization schemes of $D = 4$ Maxwell algebra, *Phys. Lett. B* 707 (2012) 292, arXiv:1111.3598 [math-ph].
- [19] J.A. de Azcarraga, J.M. Izquierdo, J. Lukierski, M. Woronowicz, Generalizations of Maxwell (super)algebras by the expansion method, *Nucl. Phys. B* 869 (2013) 303, arXiv:1210.1117 [hep-th].
- [20] S. Hoseinzadeh, A. Rezaei-Aghdam, 2+1 dimensional gravity from Maxwell and semi-simple extension of the Poincare gauge symmetric models, arXiv:1402.0320 [hep-th].
- [21] J.A. de Azcarraga, J.M. Izquierdo, Minimal $D = 4$ supergravity from the superMaxwell algebra, arXiv:1403.4128 [hep-th].
- [22] M.B. Green, Supertranslations, superstrings and Chern–Simons forms, *Phys. Lett. B* 223 (1989) 157.